Interactive Light Scattering with Principal-Ordinate Propagation: Supplementary Materials

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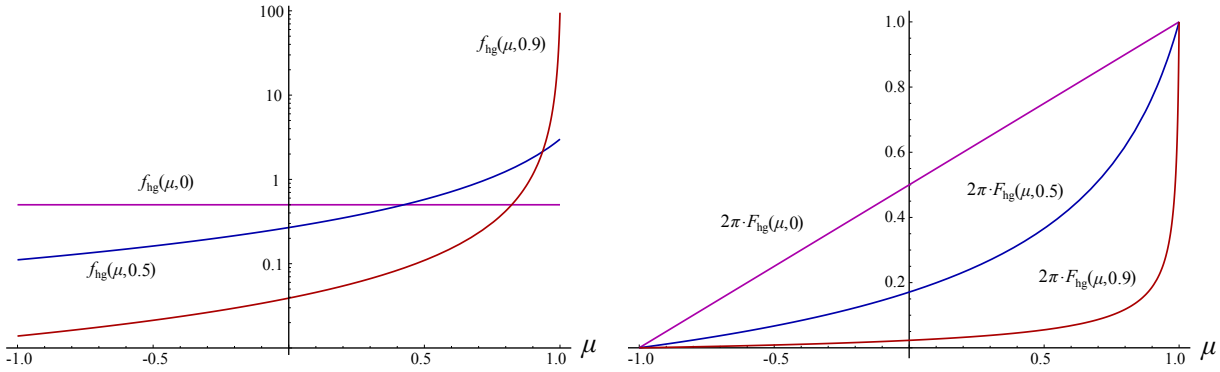

Figure 1: Plots of f_{hg} (left) and F_{hg} (right) for parameter g values of 0, 0.5 and 0.9. Please note that the plots of f_{hg} have a logarithmic scale and that we multiplied the values of F_{hg} by 2π to scale the plots to the $[0,1]$ range.

1 Theoretical Analysis

In the following text we would like to expand the discussion of our propagation scheme and justify its main steps. We still rely on the reader being familiar with the paper.

In Sec. 1.1 we recall the Henyey-Greenstein function and summarize its main properties utilized by our method. Sec. 1.2 provides a proof of energy conservation of our propagation scheme and its convergence. Finally, Sec. 1.3 demonstrates how we obtained our description of scattering with the help of the path integral framework.

1.1 Basic Definitions

The Henyey-Greenstein function (HGF hereinafter), or Henyey-Greenstein distribution, is a parametric function first introduced in [2]. One of its possible definitions is the spherical variant

$$f_{hg}(\mu,g) = \frac{1}{4\pi} \cdot \frac{1-g^2}{(1+g^2-2g\mu)^{3/2}} \tag{1}$$

with $\mu = \cos(\theta)$, where $\theta \in [0,\pi]$ is the polar (zenith) angle. HGF is rotationally symmetric around the zenith axis (i. e., its partial derivative in respect to the azimuthal angle $\phi \in [0,2\pi]$ is 0). Also, as HGF is always non-negative and has a unit integral on a sphere, it is a valid spherical probability distribution.

HGF has proven useful for approximating scattering phase functions, since it represents the preferred directionality of scattered distributions by only one scalar parameter, $g \in [-1,1]$. HGF is also an identity function, because of its following property:

$$g = 2\pi \int_0^{\pi} f_{hg}(\cos(\theta),g) \cdot \cos(\theta) \cdot \sin(\theta)\, \mathrm{d}\theta.$$

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The *anisotropy* parameter g can therefore be intuitively understood as an average cosine of the zenith angle θ across a sphere. Using HGF as a phase function implies that positive values of g correspond to dominantly forward scattering, negative g's represent backward scattering, and $g = 0$ defines the constant (isotropic) phase function.

In our work we utilize HGF to both model the scattering phase function and represent the directional distribution of radiance in a propagation volume cell. The latter on one hand limits us to unimodal distributions of radiance (please refer to the discussion in the paper for details). On the other hand, evaluating energy flow between neighbouring cells becomes much easier this way, thanks to the analytical integrability of HGF. Calculating the integral of Eq. 1 means evaluating its cumulative distribution function

$$F_{hg}(\mu,g) = \int_{-1}^{\mu} f_{hg}(\mu',g)\, d\mu',$$

for which an analytic solution exists:

$$F_{hg}(\mu,g) = \frac{1-g^2}{4\pi g} \cdot \left(\frac{1}{(1+g^2-2g\mu)^{1/2}} - \frac{1}{1+g} \right). \tag{2}$$

More precisely, Eq. 2 returns a circular integral of HGF, which therefore needs to be multiplied by the azimuth angle ϕ to obtain the spherical integral (2π for the whole sphere). Please refer to Fig. 1 for sample plots of Eq. 1 and Eq. 2.

Another useful property of HGF we make use of is its *self-convolution* property [3]. This means that the distribution resulting from i successive convolutions of f_{hg} is again a HGF parametrized by the product of the convolved distributions' parameters:

$$\underbrace{f_{hg}(g_1) * \cdots * f_{hg}(g_i)}_{i} = f_{hg}\Big(\prod_{j=1}^{i} g_j\Big). \tag{3}$$

However, this requires $\forall j \in [1..i] : \; g_j \in [0,1]$. If this is fulfilled we can see that the resulting HGF will have a lower g that any of the convolved distributions. This agrees with the intuition that applying a convolution to a distribution should result in distributions with decreasing anisotropy.

1.2 Proof of convergence

An important concern in an iterative algorithm such as the proposed one is to ensure its convergence to an energy equilibrium in a finite number of iterations. Here we demonstrate this is indeed the case.

We first recall the equation used by our method to compute the radiance contribution $\Delta L_{src\to dst}$ from cell *src* to cell *dst*:

$$\begin{aligned}\Delta L_{src\to dst} &= L_{src}\cdot T_{src\to dst}\cdot|\phi_1-\phi_2| \\ &\quad\cdot\left(F_{\mathrm{hg}}(\cos\theta_1,a_{src})-F_{\mathrm{hg}}(\cos\theta_2,a_{src})\right).\end{aligned} \tag{4}$$

We need to demonstrate that this formula does not distribute more radiant energy to the six neighbouring *dst* cells that the source cell *src* contains. For this, let us first consider an empty propagation domain, i. e., $\sigma_\mathrm{a}=\sigma_\mathrm{s}=0$ in the entire considered portion of the scene. Then the transmittance $T_{src\to dst}=1$ and Eq. 4 simplifies to

$$\begin{aligned}\Delta L_{src\to dst} &= L_{src}\cdot|\phi_1-\phi_2| \\ &\quad\cdot\left(F_{\mathrm{hg}}(\cos\theta_1,a_{src})-F_{\mathrm{hg}}(\cos\theta_2,a_{src})\right).\end{aligned} \tag{5}$$

If we first consider propagating the energy from *src* to the entire sphere, then $|\phi_1-\phi_2|=2\pi$, $\theta_1=0$ and $\theta_2=\pi$ and the resulting propagated energy will be

$$\begin{aligned}\Delta L_{src\to\Omega_{4\pi}} &= L_{src}\cdot 2\pi\cdot\left(F_{\mathrm{hg}}(1,a_{src})-F_{\mathrm{hg}}(-1,a_{src})\right) \\ &= L_{src}\cdot 2\pi\cdot\left(\frac{1}{2\pi}-0\right) \\ &= L_{src}\end{aligned} \tag{6}$$

as F_{hg} peaks at $1/2\pi$ regardless of its anisotropy parameter (see Fig. 1, right). At this point we need to ensure that our parametrization of the solid angles $\Omega_{src\to dst}$ subtended by the six neighbouring cells divides the sphere of directions into 6 mutually exclusive solid angles without any gaps. Using our approximate parametrization in terms of polar angles (also refer to the depiction in Fig. 2)

$$(\theta_1,\theta_2,|\phi_1-\phi_2|)=\begin{cases}(0,\frac{\pi}{4},2\pi) & dst \text{ in front of } src\\ (\frac{\pi}{4},\frac{3\pi}{4},\frac{\pi}{2}) & dst \text{ next to } src\\ (\frac{3\pi}{4},\pi,2\pi) & dst \text{ behind } src\end{cases} \tag{7}$$

we can rewrite Eq. 6 as (leaving out the anisotropy parameters a_{src} of F_{hg} for better clarity)

$$\begin{aligned}\Delta L_{src\to\Omega_{4\pi}} &= L_{src}\cdot 2\pi\cdot\left(F_{\mathrm{hg}}(\cos 0)-F_{\mathrm{hg}}(\cos\pi)\right) \\ &= L_{src}\cdot 2\pi\cdot\left(F_{\mathrm{hg}}(\cos 0)-F_{\mathrm{hg}}(\cos\frac{\pi}{4})\right)+ \\ &\quad+4\cdot L_{src}\cdot\frac{\pi}{2}\cdot\left(F_{\mathrm{hg}}(\cos\frac{\pi}{4})-F_{\mathrm{hg}}(\cos\frac{3\pi}{4})\right)+ \\ &\quad+L_{src}\cdot 2\pi\cdot\left(F_{\mathrm{hg}}(\cos\frac{3\pi}{4})-F_{\mathrm{hg}}(\cos\pi)\right).\end{aligned}$$

We can now easily see that this decomposition corresponds to an expansion of the following equation

$$\Delta L_{src\to\Omega_{4\pi}} = L_{src} = \sum_{dst}\Delta L_{src\to dst} \tag{8}$$

when the corresponding parameters for each of the six *dst* cells are used according to Eq. 7. It now becomes obvious that this is an exactly mirrored version of the energy accumulation equation as defined in the paper:

$$L_{dst} = \sum_{src}\Delta L_{src\to dst}. \tag{9}$$

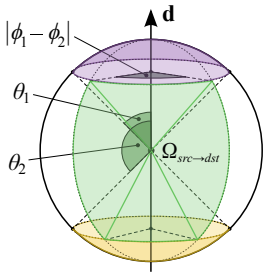

Figure 2: Our approximate parametrization of the solid angles subtended by neighbouring cells in terms of spherical coordinates. (The polar angles parametrizing the green solid angle are labelled.)

We can now see that propagating radiant energy either *from* a cell (Eq. 8) or gathering *into* a cell (Eq. 9) by using $\Delta L_{src\to dst}$ as defined in Eq. 5 conserves energy perfectly. Therefore, unless a cyclic propagation domain is used, even at this point our algorithm will eventually converge, since a small portion of the energy contained in the propagation volume will dissipate away from its outermost cells in every iteration. Furthermore, in an absorbing medium ($\sigma_\mathrm{a}>0$) the transmittance $T_{src\to dst}<1$ in Eq. 4 will lead to an additional attenuation of the propagated energy, increasing the rate of convergence.

1.3 Deriving radiance anisotropy contribution

In spite of our approach using some heuristic steps to achieve better performance in exchange for bias, it is still based on the underlying mathematical description of light propagation in participating media. We would now like to show the connection between the radiative transport equation and the way our method treats scattering as an iterative convolution of radiance distributions and the medium phase function.

In the first part of this section we follow the argumentation of Premože et al. [4, 5] in their work on formulating light scattering within the path integral framework. Recalling the time-dependent radiative transport equation [1]

$$\begin{aligned}&\left(\frac{\partial}{\partial s}+\omega\cdot\nabla+\sigma_\mathrm{t}\right)\cdot L(s,\mathbf{x},\omega)= \\ &\sigma_\mathrm{s}\int_{4\pi}L(s,\mathbf{x},\omega')f(\omega',\omega)\,\mathrm{d}\omega'+L_0(\mathbf{x},\omega)\end{aligned} \tag{10}$$

one can already see that the change of the directional radiance distribution L at the position $\mathbf{x}$ is a spherical convolution with a general phase function f. Here the dependence on time t is expressed in terms of travelled distance $s=ct$, c being the speed of light. Additionally, $\sigma_\mathrm{t}=\sigma_\mathrm{a}+\sigma_\mathrm{s}$ is the extinction coefficient and L_0 is the (initial) source radiance distribution.

Equation 10 can be expressed as a convolution of L_0 with the Green's function G:

$$L(s,\mathbf{x},\omega)=\int G(s,\mathbf{x},\mathbf{x}',\omega,\omega')L_0(\mathbf{x}',\omega')\,\mathrm{d}\mathbf{x}'\mathrm{d}\omega'.$$

The Green's function $G(s,\mathbf{x},\mathbf{x}',\omega,\omega')$, also called the *evolution operator* or *point spread function*, represents the radiance distribution at $(\mathbf{x},\omega)$ and time s originating at $(\mathbf{x}',\omega')$ and zero time. Mathematically, G is the solution of the homogeneous version of Eq. 10 when the initial boundary condition is expressed as

$$G(s=0,\mathbf{x},\mathbf{x}',\omega,\omega')=\delta(\mathbf{x}-\mathbf{x}')\delta(\omega-\omega'),$$

where δ is the Dirac function. Premože et al. [4,5] have shown that G can be used to formulate light propagation in non-scattering and single-scattering media, and described an approximate analytical solution to general multiple scattering formulated via G.

At this point we can use the expansion of G into a superposition of "partial" Green's functions. Consider a propagation from an initial source position $\mathbf{x}_I$ to a final destination position $\mathbf{x}_F$. The propagator $G(s,\mathbf{x}_I,\mathbf{x}_F,\omega_I,\omega_F)$ can be expressed as

$$G(s,\mathbf{x}_I,\mathbf{x}_F) = \lim_{n\to\infty} \int \cdots \int G(s/N,\mathbf{x}_I,\mathbf{x}_1)\cdots G(s/N,\mathbf{x}_{N-1},\mathbf{x}_F)\,\mathrm{d}\mathbf{x}_1\cdots\mathrm{d}\mathbf{x}_{N-1} \tag{11}$$

leaving out the directional parameters ω_I and ω_F for clarity. This expansion is called a *path integral* and its idea is to break the path into infinitesimally small segments and propagate energy within each segment separately. Now, if s is equal to the distance between $\mathbf{x}_I$ and $\mathbf{x}_F$ then all the auxiliary vertices $\mathbf{x}_1 \ldots \mathbf{x}_{N-1}$ will lie on a straight path. Then, during the propagation, the change of the directional distribution of energy will correspond to a spherical convolution with the partial propagator along each linear segment.

This is exactly the notion we utilize in our propagation scheme. Our propagation volume corresponds to a discretization of Eq. 11 where each pair of neighbouring cells is virtually connected by a linear path segment. Our propagation then approximates the change of the directional radiance distribution along this segment.

Since the radiance distributions at each cell are aligned, we can define $\omega = \omega_{src} = \omega_{dst}$. An additional assumption we describe in the paper is that the scattering coefficient σ_s be constant along the path between neighbouring cells *src* and *dst*, and we calculate it simply as an average of the σ_s at $\mathbf{x}_{src}$ and $\mathbf{x}_{dst}$. We can now define the partial propagator as

$$G(s,\mathbf{x}_{src},\mathbf{x}_{dst},\omega) = \sigma_s \int_s \int_{4\pi} f(\omega,\omega')\,\mathrm{d}\omega'\,\mathrm{d}s'$$

where $s = |\mathbf{x}_{dst} - \mathbf{x}_{src}|$ and f is a general phase function. However, since we describe the directional distribution of radiance at *src* by a HGF parametrized by anisotropy parameter a_{src}, and we also model the medium phase function by a HGF with anisotropy g, the change of the directional radiance distribution can be expressed as

$$\Delta a_{src\to dst} = a_{src} \cdot g^{\sigma_s \cdot s}, \tag{12}$$

which we obtain by utilizing the self-convolution property of HGF (Eq. 3). Eq. 12 is exactly what our method uses to describe the radiance anisotropy contribution.

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