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Spectral Ray Differentials
Supplemental Material – Derivation

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1. Notes

Note that the partial differentials of $\mathbf{d}$ and $\mathbf{n}$ are vectors. All other variables and differentials are scalar values.

The ratio of the indices of refraction η is now dependent on the wavelength. It is defined by $\eta = \mathrm{n}_0(\lambda)/\mathrm{n}_1(\lambda)$ where $\mathrm{n}_0(\lambda)$ and $\mathrm{n}_1(\lambda)$ are the index of refraction (IOR) of the medium through which the ray travelled and the IOR of the medium that the newly generated ray will enter. For more details on how to compute the partial differential η w.r.t. wavelength please refer to the paper.

2. Definition

Compute the new direction of the refracted ray $\mathbf{R} = (\mathbf{p}, \mathbf{d})$ with respect to index of refraction η:

$$\mathbf{d}' = \eta\mathbf{d} - \mu\mathbf{n}$$

$$\mu = \eta\,(\mathbf{d}\cdot\mathbf{n}) + \omega$$

$$\omega = \sqrt{1 - \eta^2 + \eta^2\,(\mathbf{d}\cdot\mathbf{n})^2}$$

Let $\theta = \mathbf{d}\cdot\mathbf{n}$, then:

$$\mathbf{d}' = \eta\mathbf{d} - (\eta\theta + \omega)\,\mathbf{n} = \eta\mathbf{d} - \left(\eta\theta + \sqrt{1-\eta^2+\eta^2\theta^2}\right)\mathbf{n}$$

To compute the partial differential of θ, the product rule is applied to two vectors, so a dot product needs to be used which, again, will result in a scalar value:

$$\frac{\partial\theta}{\partial\lambda} = \frac{\partial\mathbf{d}}{\partial\lambda}\cdot\mathbf{n} + \mathbf{d}\cdot\frac{\partial\mathbf{n}}{\partial\lambda}$$

3. Derivation

Seek the partial differential of $\mathbf{d}'$ w.r.t λ:

$$\frac{\partial\mathbf{d}'}{\partial\lambda} = \frac{\partial(\eta\mathbf{d})}{\partial\lambda} - \frac{\partial(\mu\mathbf{n})}{\partial\lambda}$$

Solve each term separately, by applying the product rule:

$$\frac{\partial\mathbf{d}'}{\partial\lambda} = \frac{\partial\eta}{\partial\lambda}\mathbf{d} + \eta\frac{\partial\mathbf{d}}{\partial\lambda} - \left(\frac{\partial\mu}{\partial\lambda}\mathbf{n} + \mu\frac{\partial\mathbf{n}}{\partial\lambda}\right)$$

Compute the partial differential $\frac{\partial\mu}{\partial\lambda}$:

$$\frac{\partial\mu}{\partial\lambda} = \frac{\partial(\eta\theta)}{\partial\lambda} + \frac{\partial\omega}{\partial\lambda} = \frac{\partial\eta}{\partial\lambda}\theta + \eta\frac{\partial\theta}{\partial\lambda} + \frac{\partial\omega}{\partial\lambda}$$

The partial differential $\frac{\partial\omega}{\partial\lambda}$ is computed using the chain rule:

$$\frac{\partial\omega}{\partial\lambda} = \frac{1}{2}\frac{1}{\sqrt{1-\eta^2+\eta^2\theta^2}}\frac{\partial(1-\eta^2+\eta^2\theta^2)}{\partial\lambda} = \frac{1}{2}\omega^{-\frac{1}{2}}\frac{\partial(1-\eta^2+\eta^2\theta^2)}{\partial\lambda}$$

$$\frac{\partial(1-\eta^2+\eta^2\theta^2)}{\partial\lambda} = -2\eta\frac{\partial\eta}{\partial\lambda} + \frac{\partial(\eta\theta)^2}{\partial\lambda}$$

$$\frac{\partial(\eta\theta)^2}{\partial\lambda} = 2\eta\theta\left(\frac{\partial\eta}{\partial\lambda}\theta + \eta\frac{\partial\theta}{\partial\lambda}\right) = 2\eta\frac{\partial\eta}{\partial\lambda}\theta^2 + 2\eta^2\theta\frac{\partial\theta}{\partial\lambda}$$

$$\frac{\partial\omega}{\partial\lambda} = \frac{-2\eta\frac{\partial\eta}{\partial\lambda} + 2\eta\frac{\partial\eta}{\partial\lambda}\theta^2 + 2\eta^2\theta\frac{\partial\theta}{\partial\lambda}}{2\omega} = \frac{-\eta\frac{\partial\eta}{\partial\lambda} + \eta\frac{\partial\eta}{\partial\lambda}\theta^2 + \eta^2\theta\frac{\partial\theta}{\partial\lambda}}{\omega}$$

4. Final

Putting everything together, we get:

$$\frac{\partial\mathbf{d}'}{\partial\lambda} = \frac{\partial\eta}{\partial\lambda}\mathbf{d} + \eta\frac{\partial\mathbf{d}}{\partial\lambda} - \frac{\partial\mu}{\partial\lambda}\mathbf{n} - \mu\frac{\partial\mathbf{n}}{\partial\lambda}$$

$$\frac{\partial\mu}{\partial\lambda} = \frac{\partial\eta}{\partial\lambda}\theta + \eta\frac{\partial\theta}{\partial\lambda} + \frac{-\eta\frac{\partial\eta}{\partial\lambda} + \eta\frac{\partial\eta}{\partial\lambda}\theta^2 + \eta^2\theta\frac{\partial\theta}{\partial\lambda}}{\omega}$$

$$\frac{\partial\mathbf{d}'}{\partial\lambda} = \frac{\partial\eta}{\partial\lambda}\mathbf{d} + \eta\frac{\partial\mathbf{d}}{\partial\lambda} - \left(\frac{\partial\eta}{\partial\lambda}\theta + \eta\frac{\partial\theta}{\partial\lambda} + \frac{\partial\omega}{\partial\lambda}\right)\mathbf{n} - \mu\frac{\partial\mathbf{n}}{\partial\lambda}$$